

*On the Electric Resistance to the Motion of a Charged Conducting
Sphere in Free Space or in a Field of Force.*

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In his investigations on the mode of decay of vibratory motion,* Prof. Love has demonstrated the great importance of considering the effect of the medium. In the particular case of a pendulum vibrating in air, it appears that the customary interpretation of the reaction of the medium on the pendulum, as an addition to the effective inertia along with a viscous term, is a good approximation under certain conditions. But in order to get an exact idea of what goes on, and justify the usual interpretation, it is necessary to examine in detail the motion of the medium, for it is only by this means that we can prescribe the conditions under which the usual interpretation is valid, and determine when this interpretation fails.

Turning to the case of electrical vibrations, it appears to me that Prof. Love's results have a very important bearing on the questions of electrical inertia and electrical damping. The present paper is an attempt to apply his method to some questions connected with the motion of an electrified spherical conductor.

In the paper referred to, a discussion is given of the vibration set up by a fixed conducting sphere when the electricity is initially distributed with a surface density proportional to the first zonal harmonic P_1 . The total charge on the sphere is zero, and it is easy to verify that the resultant mechanical force on the sphere is nil.

If however we suppose that the sphere has a resultant charge, so that we have a uniform surface density superposed on the surface density proportional to P_1 , then, since vibrations of zero order cannot be propagated, the determination of the vibrations of order unity is exactly the same as in the case discussed by Prof. Love. I find however that in this case there is a resultant mechanical force on the sphere in the direction of z , the axis of the harmonic P_1 . Thus a mechanical force must be applied to the sphere in order to keep it at rest, and this force vanishes only when the vibrations have subsided. If the constraint is not applied the sphere must move.

The motion of a charged conductor in a uniform field of electrical force is

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a matter of great importance. We shall begin by considering the following problem :—A perfectly conducting sphere of radius a , possessing a charge e is placed in a uniform field of electrical force F , and initially constraints exist which keep the sphere at rest with a uniform surface density. The constraints then cease to act, and we have to determine what ensues.

If z be the direction of F , the effect of F is (1) to make the sphere move in that direction, (2) to tend to establish an additional distribution on the surface proportional to the first zonal harmonic P_1 . Vibrations of order unity thus become possible, and we shall confine our attention at first to the state of matters so long as it is permissible to neglect squares of the displacement; that is to say, if ξ is the displacement of the centre of the sphere, we shall neglect squares of ξ/a . As far as possible I shall follow Prof. Love's notation, taking the initial position of the centre of the sphere as a fixed origin of reference. I shall use x_0, y_0, z_0 for the co-ordinates of a point referred to this origin.

Initially we have at all points outside the sphere

$$(X, Y, Z) = e \left(\frac{x_0}{r_0^3}, \frac{y_0}{r_0^3}, \frac{z_0}{r_0^3} \right) + (0, 0, 1) F; \quad (1)$$

$$(\alpha, \beta, \gamma) = 0.$$

This state cannot continue. At a subsequent instant, when the centre of the sphere is at ξ along the z_0 axis, let the state of the medium outside the sphere be given by

$$(X, Y, Z) = e \left(\frac{x_0}{r_0^3}, \frac{y_0}{r_0^3}, \frac{z_0}{r_0^3} \right) + (0, 0, 1) F$$

$$+ c \left(\frac{\partial^2}{\partial x_0 \partial z_0}, \frac{\partial^2}{\partial y_0 \partial z_0}, -\frac{\partial^2}{\partial x_0^2}, -\frac{\partial^2}{\partial y_0^2} \right) \frac{\chi(ct - r_0)}{r_0}; \quad (2)$$

$$(\alpha, \beta, \gamma) = \left(\frac{\partial^2}{\partial y_0 \partial t}, -\frac{\partial^2}{\partial x_0 \partial t}, 0 \right) \frac{\chi(ct - r_0)}{r_0}.$$

The conditions that must be satisfied at the surface $r_0 = ct + a$, which separates the disturbed and undisturbed portions of the medium, require that

$$\chi = 0 \quad \text{and} \quad \chi' = 0 \quad \text{when} \quad r_0 = ct + a.$$

In order to apply the condition that the tangential component of electric force at the surface of the sphere should vanish, it is convenient to express the state of the medium with reference to a new fixed origin which at the instant coincides with the centre of the sphere. If x, y, z, r refer to this origin, and we may premise the result, which appears in the analysis, that χ and $e\xi/c$ are small quantities proportional to F , then we get approximately

$$\begin{aligned}
(X, Y, Z) &= e \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right) - e \xi \left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2}{r^5} - \frac{1}{r^3} \right) \\
&\quad + c \left(\frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial y \partial z}, -\frac{\partial^2}{\partial x^2}, -\frac{\partial^2}{\partial y^2} \right) \chi \frac{(ct-r)}{r} + (0, 0, 1) F; \quad (3) \\
(\alpha, \beta, \gamma) &= \left(\frac{\partial^2}{\partial y \partial t}, -\frac{\partial^2}{\partial x \partial t}, 0 \right) \chi \frac{(ct-r)}{r}.
\end{aligned}$$

This may be written in the form

$$\begin{aligned}
(X, Y, Z) &= e \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right) + (0, 0, 1) \left\{ F - \frac{c}{r^3} \left(r^2 \chi'' + r \chi' + \chi - \frac{e \xi}{c} \right) \right\} \\
&\quad + c \left(\frac{xz}{r^5}, \frac{yz}{r^5}, \frac{z^2}{r^5} \right) \left\{ r^2 \chi'' + 3r \chi' + 3 \left(\chi - \frac{e \xi}{c} \right) \right\}; \quad (4) \\
(\alpha, \beta, \gamma) &= c \left(\frac{y}{r^3}, -\frac{x}{r^3}, 0 \right) (r \chi'' + \chi').
\end{aligned}$$

From these we see that the tangential electric force vanishes at the surface of the sphere if

$$a^2 \chi'' (ct-a) + a \chi' (ct-a) + \chi (ct-a) - \frac{e \xi}{c} = \frac{a^3 F}{c}. \quad (5)$$

The surface density is given by

$$\sigma = \frac{1}{4\pi} \left\{ \frac{e}{a^2} + \frac{c P_1}{a^3} \left(a^2 \chi'' + 3a \chi' + 3\chi - \frac{3e \xi}{c} \right) \right\}.$$

Or, in virtue of (5),

$$\sigma = \frac{1}{4\pi} \left\{ \frac{e}{a^2} + \frac{c P_1}{a^3} \left(\frac{3a^3 F}{c} - 2a^2 \chi'' \right) \right\}. \quad (6)$$

The Z component of electric force

$$= \frac{e P_1}{a^2} + c \frac{P_1^2}{a^3} \left(\frac{3a^3 F}{c} - 2a^2 \chi'' \right). \quad (7)$$

It is easy to see that the resultant of all the mechanical forces when integrated over the sphere arises solely from the Z component of electric force.

The mechanical force in the direction of z

$$= \frac{1}{2} \int \sigma Z dS = e F - \frac{2}{3} \frac{e c}{a} \chi'' (ct-a). \quad (8)$$

Thus if m is the ordinary mass of the sphere, the equation of motion is

$$m \ddot{\xi} = e F - \frac{2}{3} \frac{e}{a} c \chi'' (ct-a),$$

or

$$m \ddot{\xi} + \frac{2}{3} \frac{e}{a} c \chi'' (ct-a) = e F. \quad (9)$$

Equations (5) and (9), along with the conditions $\chi(ct-r) = 0$, $\chi'(ct-r) = 0$, when $r = ct + a$, and $\zeta = 0$, $\dot{\zeta} = 0$, when $t = 0$, determine the motion.

It is of interest to note that equations (5) and (9) may be regarded as the equations of motion of a dynamical system with two degrees of freedom.

Putting $\chi = \frac{e}{c}(\phi + \zeta)$, the equations may be written

$$m'\ddot{\phi} + m'\ddot{\zeta} + \frac{c}{a}m'\dot{\phi} + \frac{c}{a}m'\dot{\zeta} + \frac{c^2}{a^2}m'\phi = \frac{2}{3}eF;$$

$$(m+m')\ddot{\zeta} + m'\ddot{\phi} = eF,$$

where

$$m' = \frac{2}{3} \frac{e^2}{ac^2}.$$

Thus the kinetic energy is

$$T = \frac{1}{2}m'(\dot{\phi} + \dot{\zeta})^2 + \frac{1}{2}m\dot{\zeta}^2 + \frac{1}{4}\frac{c}{a}m'(\zeta\dot{\phi} - \phi\dot{\zeta}),$$

the dissipation function

$$F = \frac{1}{2}\frac{c}{a}m'\dot{\phi}(\dot{\phi} + \dot{\zeta}),$$

and the potential energy

$$V = \frac{1}{2}\frac{c^2}{a^2}m'\phi^2.$$

If we put $\xi = ct - a$ equations (5) and (9) may be written

$$\frac{d^2\chi}{d\xi^2} + \frac{1}{a}\frac{d\chi}{d\xi} + \frac{1}{a^2}\left(\chi - \frac{e\xi}{c}\right) = \frac{aF}{c}; \quad (10)$$

$$m\frac{d^2\zeta}{d\xi^2} + \frac{2}{3}\frac{e}{ac}\frac{d^2\chi}{d\xi^2} = \frac{e}{c^2}F, \quad (11)$$

with the conditions $\zeta = 0$, $\dot{\zeta} = 0$, when $t = 0$ and $\chi(-a) = 0$, $\chi'(-a) = 0$.

The solution of (11) is

$$m\zeta + \frac{2}{3}\frac{e}{ac}\chi = \frac{1}{2}\frac{eF}{c^2}\xi^2 + B\xi + D,$$

where B and D are constants of integration. Applying the initial conditions, we get

$$m\zeta + \frac{2}{3}\frac{e}{ac}\chi = \frac{1}{2}\frac{e}{c^2}F(\xi + a)^2. \quad (12)$$

Substituting in (10), and putting $\frac{2}{3}\frac{e^2}{ac^2} = m'$, we get

$$\frac{d^2\chi}{d\xi^2} + \frac{1}{a}\frac{d\chi}{d\xi} + \frac{1}{a^2}\left(1 + \frac{m'}{m}\right)\chi = \frac{aF}{c}\left\{1 + \frac{3}{4}\frac{m'}{m}\frac{(\xi + a)^2}{a^2}\right\}.$$

The solution of this equation satisfying the initial conditions is

$$\chi = Ae^{-\frac{\xi+a}{2a}} \sin \left\{ \left(3 + \frac{4m'}{m} \right)^{\frac{1}{2}} \frac{\xi+a}{2a} + \epsilon \right\} + A'(\xi+a)^2 + B'(\xi+a) + D', \quad (13)$$

where

$$\begin{aligned} A' &= \frac{3}{4} \frac{m'}{m+m'} \frac{\alpha F}{C}, & B' &= \frac{3}{2} \frac{mm'}{(m+m')^2} \frac{\alpha^2 F}{C}, \\ D' &= \frac{m(2m^2+4mm'-m'^2)}{2(m+m')^3} \frac{\alpha^3 F}{C}, & A \sin \epsilon &= -D', \\ & & \left(3 + \frac{4m'}{m}\right)^{\frac{1}{2}} A \cos \epsilon &= -(D' + 2\alpha B'). \end{aligned}$$

Hence

$$\begin{aligned} \chi(CT-r) &= Ae^{-(CT-r+a)/2a} \sin \left\{ \left(3 + \frac{4m'}{m}\right)^{\frac{1}{2}} \frac{CT-r+a}{2a} + \epsilon \right\} \\ &\quad + A'(CT-r+a)^2 + B'(CT-r+a) + D', \end{aligned} \quad (14)$$

and

$$\begin{aligned} \xi &= -\frac{2}{3} \frac{e}{maC} Ae^{-CT/2a} \sin \left\{ \left(3 + \frac{4m'}{m}\right)^{\frac{1}{2}} \frac{CT}{2a} + \epsilon \right\} \\ &\quad + \frac{1}{2} \frac{eF}{(m+m')} t^2 + \frac{eFm'}{(m+m')^2} \frac{at}{C} - \frac{1}{3} \frac{eF(2m^2+4mm'-m'^2)}{(m+m')^3} \frac{\alpha^2}{C^2}. \end{aligned} \quad (15)$$

These results hold as long as ζ/a is small. We may observe that the initial displacement expressed by the damped harmonic part of the motion is equal and opposite to the displacement expressed by the non-periodic portion. After one complete vibration the amplitude of the vibratory part is $e^{-2\pi/(3+4m'/m)^{\frac{1}{2}}}$ of its initial value. Thus if the displacement at time t is small, the vibratory part of the motion may have practically become insignificant before the equations become invalid. Since the decay is exponential, this will be secured even when ct/a is only moderately great, while the condition that ζ/a is small can be secured by making F small.

In these circumstances the displacement of the sphere is adequately represented by

$$\zeta = \frac{1}{2} \frac{eF}{(m+m')} t^2 + \frac{eFm'}{(m+m')^2} \frac{a}{C} t - \frac{1}{3} \frac{eF(2m^2+4mm'-m'^2)}{(m+m')^3} \frac{\alpha^2}{C^2}, \quad (16)$$

and

$$\begin{aligned} \chi(CT-r) &= \frac{3}{4} \frac{m'}{m+m'} \frac{\alpha F}{C} (CT-r+a)^2 - \frac{3}{2} \frac{mm'}{(m+m')^2} \frac{\alpha^2 F}{C} (CT-r+a) \\ &\quad + \frac{m(2m^2+4mm'-m'^2)}{2(m+m')^3} \frac{\alpha^3 F}{C}, \end{aligned} \quad (17)$$

within a certain region.

We may readily verify that the state of the field within this region is given by

$$\begin{aligned} (X, Y, Z) &= e \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right) + (0, 0, 1) \left\{ F - \frac{C}{r^3} (L + A'r^2) \right\} \\ &\quad + C \left\{ \frac{xx}{r^5}, \frac{yz}{r^5}, \frac{z^2}{r^5} \right\} \{ 3L + A'r^2 \}; \end{aligned} \quad (18)^*$$

* At greater distances the damped harmonic train would have to be retained were the wave-train not limited.

$$(\alpha, \beta, \gamma) = \left(\frac{y}{r^3}, -\frac{x}{r^3}, 0 \right) \frac{e\dot{\zeta}}{c},$$

where

$$A' = \frac{3}{4} \frac{m'}{m+m'} \frac{aF}{c}, \quad L = \frac{a^3 F}{c} - a^2 A',$$

the origin being the centre of the sphere at time t . The surface density is

$$\sigma = \frac{1}{4\pi} \left\{ \frac{e}{a^2} + 3P_1 \frac{2m+m'}{2(m+m')} F \right\}. \quad (19)$$

Before discussing these results, and in case exception may be taken to the artificial nature of the initial state, I shall consider the following case. The charged sphere is placed in a uniform field and held at rest until the distribution of electricity is that appropriate to a uniform external applied field. The sphere is then let go, and we have to determine the subsequent state. The initial state is thus given by

$$\begin{aligned} (X, Y, Z) &= e \left(\frac{x_0}{r_0^3}, \frac{y_0}{r_0^3}, \frac{z_0}{r_0^3} \right) + (0, 0, 1) F \\ &\quad + Fa^3 \left(\frac{3x_0 z_0}{r_0^5}, \frac{3y_0 z_0}{r_0^5}, \frac{3z_0^2}{r_0^5} - \frac{1}{r_0^3} \right); \\ (\alpha, \beta, \gamma) &= 0. \end{aligned}$$

If we proceed in the same manner as before, it is fairly obvious that the differential equations for ζ and χ are the same as before, viz., (10) and (11). The initial conditions are now $\zeta = 0$, $\dot{\zeta} = 0$, when $t = 0$ and $\chi(-a) = Fa^3/c$, $\chi'(-a) = 0$.

Under the same conditions as before, I find that when the vibratory part has become negligible,

$$\begin{aligned} \zeta &= \frac{1}{2} \frac{eF}{(m+m')} t^2 + \frac{eFm'}{(m+m')^2} \frac{a}{c} t - \frac{1}{3} \frac{eF(2m^2+4mm'-m'^2)}{(m+m')^3} \frac{a^2}{c^2} \\ &\quad + \frac{2}{3} \frac{eF}{(m+m')} \frac{a^2}{c^2}, \end{aligned} \quad (20)$$

and

$$\begin{aligned} \chi(Ct-r) &= \frac{3}{4} \frac{m'}{(m+m')} \frac{aF}{c} (Ct-r+a)^2 - \frac{3}{2} \frac{mm'}{(m+m')^2} \frac{a^2 F}{c} (Ct-r+a) \\ &\quad + \frac{m(2m^2+4mm'-m'^2)}{2(m+m')^3} \frac{a^3 F}{c} + \frac{m'}{(m+m')} \frac{a^3 F}{c}. \end{aligned} \quad (21)$$

The equations (20) and (21) differ from (16) and (17) only in the constant terms. I have also verified that (20) and (21) give the same state of the medium (18) and the same surface density (19).

Equations (16) and (20) both give

$$\dot{\zeta} = \frac{eF}{(m+m')} t + \frac{eFm'}{(m+m')^2} \frac{a}{c}; \quad \ddot{\zeta} = \frac{eF}{m+m'}.$$

The sphere thus arrives at the position given by (16) or (20), as if the equation of motion had been

$$(m+m')\ddot{\xi} = eF, \quad (22)$$

and the initial velocity had been $\frac{eFm'}{(m+m')^2c}$, and the initial position $-\frac{1}{3}\frac{eF(2m^2+4mm'-m'^2)}{(m+m')^2}\frac{a^2}{c^2}$ in the case (16), and $\frac{eFm'^2}{(m+m')^3}\frac{a^2}{c^2}$ in the case (20)

This seems to me the only interpretation which can be put on the equations.

It thus appears that the effect of the medium is to contribute m' , i.e., $\frac{2}{3}\frac{e^2}{ac^2}$

to the inertia in the equation of motion, and so far this agrees with the usual result. No other term occurs in the equation of motion (22), and the other effects are a permanent contribution to the velocity and a permanent contribution to the displacement. It thus appears that when the state represented by (16)—(19) is reached no damping action of the medium takes place. This is supported by equations (18) and (19), which show that a surface density independent of the time has been established for an origin moving with the sphere, and that the electric field in the immediate vicinity of the sphere is also independent of the time. Since ξ increases uniformly with time, the magnetic field varies. The only difference as time goes on is that the region throughout which (18) is applicable becomes greater.

This result is remarkable, and requires further examination, as it may appear to conflict with Larmor's calculation* of the rate at which energy crosses a surface at some distance from a small charged system under constant acceleration, for his result suggests continual damping.

There are however several points to be remembered.

The expressions given by Larmor for the electric and magnetic forces apply only at great distances from the origin, and require modification in the vicinity of the small system, e.g., an electron which has a definite size.†

The rate at which energy travels outwards is calculated over a surface of large radius. It must include the rate at which the portion of energy $\frac{1}{2}m'\dot{\xi}^2$ already established is redistributed throughout space. Whether this can be regarded as lost is doubtful. Larmor points out that the flow of energy in the vicinity of the small system would be different.

The rate is calculated on the supposition that the only forces are those due

* 'Æther and Matter,' p. 226, *et. seq.*

† When the sphere has a definite radius the surface at any instant will cut the space enclosed by its surface at a short interval of time previously, and thus an expansion in powers of ξ/a can be obtained. If the charge is concentrated at a mere point, this expansion fails, and some new procedure would have to be adopted.

to the motion of the system. The rate would be modified if, as in the present case, any other field of force exists.

Thus the rate calculated by Larmor cannot here be applied to give the reaction on the small system at any instant.* There is no conflict, for the conditions assumed are not the same. If the only field is that due to the motion, the radiated energy must be supplied at the expense of the energy of the moving system, and this means damping of the motion. But when in addition there is an external field producing the motion, the external field may supply the radiated energy, and the damping action is masked. The cases are analogous to those of an ordinary vibrating mechanical system with dissipation under the influence of (1) no forces, and (2) given applied forces.

The equation $(m + m') \ddot{\xi} = eF$ is not to be taken as meaning that there is no transference of energy across the surface of the sphere, for we have seen that the constants of integration have to be given certain values which indicate an apparent initial velocity and initial displacement. A reference to the dynamical specification on p. 263 shows that the external field does work in altering the co-ordinate ϕ as well as in altering the co-ordinate ξ .

The results may also be regarded as not inconsistent with those obtained by Lorentz and others. Lorentz† shows that the damping effect is represented by a term proportional to $\dddot{\xi}$ in the equation of motion, and here when the vibrations have subsided $\dddot{\xi}$ is zero.

By eliminating χ between (5) and (9) we get an equation for ξ which is valid throughout the whole time considered. The equation is—

$$(m + m') \ddot{\xi} + \frac{ma}{c} \dddot{\xi} + \frac{ma^2}{c^2} \ddot{\xi} = eF.$$

This might be regarded as the equation of motion, but since the solution involves four arbitrary constants, we cannot complete their determination without a knowledge of the medium.

We thus conclude that if the vibratory part becomes negligible, while ξ/a is still small, the equation of motion is

$$(m + m') \ddot{\xi} = eF,$$

and the position at time t is the solution of this, as if the sphere had started with a certain initial velocity and a certain initial displacement from the original centre of the sphere.

In connection with the question of electrical inertia it is of interest to

* Cf. Sommerfeld, 'Gött. Nachrichten,' 1904, vol. 5, p. 369.

† 'Théorie Electromagnétique,' p. 124.

consider what happens if m is zero. We shall examine the case where the sphere is held fixed until the surface density appropriate to a uniform field is established, and the sphere then released.

Equations (10) and (11) now become

$$\frac{d^2\chi}{d\xi^2} + \frac{1}{a} \frac{d\chi}{d\xi} + \frac{1}{a^2} \left(\chi - \frac{e\xi}{c} \right) = \frac{aF}{c}. \quad (23)$$

$$\text{and} \quad \frac{2}{3} \frac{e}{ac} \frac{d^2\chi}{d\xi^2} = \frac{e}{c^2} F. \quad (24)$$

The initial conditions are $\xi = 0$, $\dot{\xi} = 0$, when $t = 0$, $\chi(-a) = a^3F/c$, $\chi'(-a) = 0$.

The solution of (24) is

$$\chi = \frac{3}{4} \frac{a}{c} F (\xi + a)^2 + \frac{a^3F}{c}.$$

$$\text{Hence} \quad \chi(ct-r) = \frac{3}{4} \frac{a}{c} F (ct-r+a)^2 + \frac{a^3F}{c}.$$

Substituting in (23) we get

$$\zeta = \frac{1}{2} \frac{eF}{m'} t^2 + \frac{a}{c} \frac{eF}{m'} t + \frac{eF}{m'} \frac{a^2}{c^2}, \quad \text{where} \quad m' = \frac{2}{3} \frac{e^2}{ac^2}.$$

It appears that no vibratory part is set up, but the expression for ζ does not satisfy the conditions $\zeta = 0$, $\dot{\zeta} = 0$ when $t = 0$, for we cannot make a zero without invalidating the whole thing. We must thus regard the equations as failing when $m = 0$.

The question naturally arises whether any of the conclusions arrived at would be completely invalidated by retaining squares of ζ/a . If we retained squares and higher powers of ζ/a , it is obvious that we should have to assume vibrations of the second and higher orders. In the initial stages of the motion the terms introduced by the vibrations of the n th order will be proportional to F^n . Thus by taking F small enough we can make the first order terms already investigated as good an approximation as we please. Further, the vibrations of higher order die out more rapidly the higher the order. We may therefore conclude that by taking F small, the additional terms introduced will in the initial stages be small in comparison with the first order terms, and hence the conclusions will not be substantially modified.

When the state represented by equations (18), (19), (22) is reached, a constant surface density is established, and an electric field which in the vicinity of the sphere is independent of the time. The continuance of the motion cannot, as far as I can see, set up any new damped harmonic vibration of the first order, and equations (18), (19), (22) continue to represent the

state with respect to the centre of the sphere at the instant considered, the only difference as time goes on being that the space throughout which (18) is applicable increases. We may indicate the way in which this process must fail. Vibrations of higher orders must arise, and the velocity of the sphere becomes so great that the interval of time τ from t to $t + \tau$, during which ξ/a (ξ being the displacement of the sphere from a fixed origin coinciding with the centre of the sphere at time t) is small, becomes less and less, so that the vibrations have not time to become insignificant. Whether, with the vibrations of higher order, as in the case of the first order vibrations, a state is reached when no further vibration of a particular order can be set up, is a question for further investigation.

If at a time t , while equations (14) and (15) are still valid, the external field is supposed to cease, new vibrations of the first order are set up, and the system settles down to a new state.

The procedure is much the same as before. Taking the centre of the sphere at time t_1 as a fixed origin of reference, and that instant as a new origin of time, we assume quantities θ for the displacement of the sphere and ψ ($Ct - r$) for the vibrations. Neglecting squares of θ/a , the equations are

$$\begin{aligned} a^2 \psi'' (Ct - a) + a \psi' (Ct - a) + \psi (Ct - a) - e (\theta/c) &= 0; \\ m \ddot{\theta} + \frac{2}{3} (eC/a) \psi'' (Ct - a) &= 0, \end{aligned}$$

which hold at the surface of the sphere with the initial conditions

$$\psi'(-a) = \chi'(Ct_1), \quad \psi(-a) = \chi(Ct_1) (e\xi_{t_1})/c,$$

which hold at the surface of discontinuity, and

$$\theta = 0, \quad \dot{\theta} = \dot{\xi}_{t_1},$$

the values being determined from (14) and (15).

The solution of these equations is

$$\begin{aligned} \psi(Ct - r) &= A'' e^{(Ct - r + a)/2a} \sin \left\{ \left(3 + \frac{4m'}{m} \right)^{\frac{1}{2}} \frac{Ct - r + a}{2a} + \epsilon'' \right\} \\ &+ \frac{e}{(m + m')c} B (Ct - r + a) + \frac{e}{(m + m')c} \left\{ D - \frac{am'}{(m + m')} B \right\}, \end{aligned}$$

referred to the centre of the sphere at the instant considered, and t being now reckoned from the instant t_1 .

The displacement of the sphere is given by

$$\begin{aligned} \theta &= -\frac{2}{3} \frac{e}{amC} A'' e^{-Ct/2a} \sin \left\{ \left(3 + \frac{4m'}{m} \right)^{\frac{1}{2}} \left(\frac{Ct}{2a} + \epsilon'' \right) \right\} \\ &+ \frac{CtB}{(m + m')} + \frac{D}{(m + m')} + \frac{am'}{(m + m')^2} B. \end{aligned}$$

The constants are given by

$$\begin{aligned} D &= \left\{ \chi(ct_1) - \frac{e}{c} \zeta_{t_1} \right\}^{\frac{2}{3}} \frac{e}{ac} = -(m+m') \zeta_{t_1} + m \zeta_{t_1} + \frac{2}{3} \frac{e}{ac} \chi(ct_1); \\ B &= m \left(\frac{d\zeta}{d\xi} \right)_{t_1} + \frac{2}{3} \frac{e}{ac} \left(\frac{d\chi}{d\xi} \right)_{t_1}. \\ A'' \sin \epsilon'' &= \chi(ct_1) - \frac{e\zeta_{t_1}}{c} - \frac{e}{(m+m')c} \left\{ D - \frac{am}{(m+m')} B \right\}, \\ -A'' \sin \epsilon'' + \left(3 + 4 \frac{m'}{m} \right)^{\frac{1}{2}} A'' \cos \epsilon'' &= 2a \left(\frac{d\chi}{d\xi} \right)_{t_1} - \frac{2ae}{(m+m')c} B. \end{aligned}$$

If this new set of vibrations become negligible while θ/a is still small, we get as before

$$\psi(ct-r) = \frac{eB}{(m+m')c} (ct-r+a) + \frac{e}{(m+m')c} \left\{ D - \frac{am}{(m+m')^2} B \right\},$$

and
$$\theta = \frac{B}{(m+m')} ct + \frac{D}{(m+m')} + \frac{am'B}{(m+m')^2}.$$

It is readily verified that these give for the field

$$(X, Y, Z) = e \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right), \quad (\alpha, \beta, \gamma) = e \left(\frac{y}{r^3}, -\frac{x}{r^3}, 0 \right) \frac{\dot{\theta}}{c},$$

applicable within a certain region.

This agrees with the usual result that when a constant velocity has been established, the electric field given by $e(x/r^3, y/r^3, z/r^3)$, is carried with the sphere, while the magnetic field is that due to an element of current $e\dot{\theta}/c$.

We may determine B and D from equation (12), which holds whether or not the vibrations set up in starting the system have subsided at the instant t_1 , when F is supposed to cease.

We thus obtain

$$\theta = \frac{eFt_1}{(m+m')} t - \zeta_{t_1} + \frac{1}{2} \frac{eF}{(m+m')} t_1^2 + \frac{a}{c} \frac{m'}{(m+m')^2} eFt_1.$$

Hence the velocity finally established is $eFt_1/(m+m')$, which is the velocity acquired by the system having inertia $m+m'$ acted on by the force eF for a time t_1 , so that the apparent contribution to the initial velocity (see p. 266) by the first vibrations is exactly destroyed by the second vibrations. Again writing the above equation in the form

$$\theta + \zeta_{t_1} = \frac{eFt_1}{(m+m')} t + \frac{1}{2} \frac{eF}{(m+m')} t_1^2 + \frac{a}{c} \frac{m'}{(m+m')^2} eFt_1,$$

the apparent contribution to the initial displacement produced by the first vibrations is destroyed by the second vibrations. The contribution

$\frac{1}{2} \frac{eF}{(m+m')} t_1^2$ is the displacement due to the force eF acting on $(m+m')$ for a time t_1 . The contribution on account of the velocity $\frac{a}{c} \frac{m_1}{(m+m')^2} eF$ for a time t_1 , could not of course be expected to disappear. It is the only permanent effect of the medium on the sphere which could not be accounted for by the ordinary dynamics of a particle of mass $(m+m')$ acted on by a force eF for a time t_1 , and then allowed to go on. Further, there is no loss of energy, for the velocity established is $eF t_1 / (m+m')$, and the energy of the system is $\frac{1}{2} e^2 F^2 t_1^2 / (m+m')$, i.e., $eF \cdot \frac{1}{2} eF t_1^2 / (m+m')$, and is thus the work done by the force eF acting on the particle of mass $(m+m')$ for a time t_1 .

The problem is thus brought into exceedingly close relationship with ordinary dynamics.*

The satisfaction of the energy condition emphasises what I have already said about the rate at which energy crosses a boundary far away from the sphere. It must not all be regarded as energy lost to the system, but part of it must be taken as indicating how the portion of energy $\frac{1}{2} m' v^2$ is redistributed throughout space as time goes on.

If the field applied to the charge is of a periodic character, the sphere may be supposed never to move far from its original position. Thus if ζ/a is always small, the first order terms are adequate for an indefinitely great time. We shall suppose that at time $t = 0$ the periodic force $F \cos nt$ begins to act.

Referring to (10) and (11), the equations for χ and ζ are readily seen to be

$$\frac{d^2 \chi}{d\xi^2} + \frac{1}{a} \frac{d\chi}{d\xi} + \frac{1}{a^2} \left(\chi - \frac{e\zeta}{c} \right) = \frac{aF}{c} \cos nt; \quad (25)$$

$$m \frac{d^2 \zeta}{d\xi^2} + \frac{2}{3} \frac{e}{ac} \frac{d^2 \chi}{d\xi^2} = \frac{e}{c^2} F \cos nt, \quad (26)$$

with the conditions $\zeta = 0$, $\dot{\zeta} = 0$, when $t = 0$ and $\chi(-a) = 0$, $\chi'(-a) = 0$.

The solution of (26) is

$$m\zeta + \frac{2}{3} \frac{e}{ac} \chi = -\frac{e}{n^2} F \cos nt + \frac{eF}{n^2}.$$

Hence substituting in (25) we get

$$\frac{d^2 \chi}{d\xi^2} + \frac{1}{a} \frac{d\chi}{d\xi} + \frac{1}{a^2} \left(1 + \frac{m'}{m} \right) \chi = \left(\frac{a}{c} - \frac{e^2}{a^2 c m n^2} \right) F \cos nt + \frac{e^2 F}{m n^2 a^2 c}.$$

* In agreement with this we may also observe that the dissipation function, p. 263, which may be written $\frac{1}{3} c \dot{\psi} (\dot{\psi} - e\dot{\theta}/c)$, now vanishes, since the equations give $\dot{\psi} - e\dot{\theta}/c = 0$.

As the solution of this we get

$$\chi(Ct - r) = A e^{-(Ct-r+a)/2a} \sin \left\{ \left(3 + 4 \frac{m'}{m} \right)^{\frac{1}{2}} \frac{Ct-r+a}{2a} + \epsilon \right\} + \frac{e^2 F}{(m+m')n^2 C} \\ + \left(\frac{a}{C} - \frac{e^2}{a^2 m n^2 C} \right) F \frac{ \left\{ \frac{(m+m')}{a^2 m} - \frac{n^2}{C^2} \right\} \cos \frac{n(Ct-r+a)}{C} - \frac{n}{aC} \sin \frac{n(Ct-r+a)}{C} }{ \left(\frac{(m+m')}{a^2 m} - \frac{n^2}{C^2} \right)^2 + \frac{1}{a^2} \frac{n^2}{C^2} },$$

where A and ϵ are determined by the initial conditions $\chi(-a) = 0, \chi'(-a) = 0$.

Hence we get

$$\zeta = -\frac{2}{3} \frac{e}{am} A e^{-Ct/2a} \sin \left\{ \left(3 + 4 \frac{m'}{m} \right)^{\frac{1}{2}} \frac{Ct}{2a} + \epsilon \right\} + \frac{eF}{(m+m')n^2} - \frac{eF}{mn^2} F \cos nt \\ - \frac{e}{m} \left(\frac{2}{3C^2} - \frac{m'}{ma^2 n^2} \right) F \frac{ \left\{ \frac{(m+m')}{a^2 m} - \frac{n^2}{C^2} \right\} \cos nt - \frac{n}{aC} \sin nt }{ \left\{ \frac{(m+m')}{a^2 m} - \frac{n^2}{C^2} \right\}^2 + \frac{1}{a^2} \frac{n^2}{C^2} }. \quad (27)$$

The damped harmonic part rapidly becomes negligible, and we then have

$$\zeta = \frac{e}{(m+m')n^2} F - \frac{e}{mn^2} F \cos nt \\ - \frac{e}{m} \left(\frac{2}{3C^2} - \frac{m'}{ma^2 n^2} \right) F \frac{ \left\{ \frac{(m+m')}{a^2 m} - \frac{n^2}{C^2} \right\} \cos nt - \frac{n}{aC} \sin nt }{ \left\{ \frac{(m+m')}{a^2 m} - \frac{n^2}{C^2} \right\}^2 + \frac{n^2}{a^2 C^2} }. \quad (28)$$

I find as in the former examples that the initial conditions cannot be satisfied if $m = 0$, as no damped harmonic part is set up.

It is possible to interpret (28) as a solution of the equation

$$M \ddot{\zeta} + k \zeta = eF \cos nt,$$

where, with some algebraic reduction, it is found that,

$$M = \frac{ (m+m') - \frac{1}{3} (m+m') \frac{a^2 n^2}{C^2} - \frac{1}{3} m \frac{a^4 n^4}{C^4} }{ 1 + \frac{1}{3} \frac{a^2 n^2}{C^2} + \frac{1}{9} \frac{a^4 n^4}{C^4} }, \quad k = \frac{ \left(m' - \frac{2}{3} m \frac{a^2 n^2}{C^2} \right) \frac{a}{C} }{ 1 + \frac{1}{3} \frac{a^2 n^2}{C^2} + \frac{1}{9} \frac{a^4 n^4}{C^4} }.$$

We cannot make a zero, since the process assumes that ζ/a is small, but if the frequency is so small that $a^2 n^2 / C^2$ may be neglected, we get approximately $M = m + m', k = \frac{2}{3} (e^2 / C^3)$. This agrees with Lorentz's* result for an electron. It also agrees with the first term in M. Abraham's† result for a Heaviside ellipsoid when the velocity is small.

* 'Ency. d. Math. Wiss.,' vol. 5, Part 2, p. 190.

† 'Electrician,' pp. 868, 869, September 16, 1904.

By eliminating χ between (25) and (26) we get

$$(m+m')\ddot{\zeta} + \frac{ma}{c}\ddot{\zeta} + m\frac{a^2}{c^2}\ddot{\zeta} = eF\left\{\cos nt - \frac{na}{c}\sin nt - \frac{1}{3}\frac{n^2a^2}{c^2}\cos nt\right\},$$

applicable from the initial instant. For the complete determination of the arbitrary constants the initial values of ζ and $\dot{\zeta}$ are not sufficient.

These examples seem to me to emphasise the necessity of considering the equations for the sphere and the medium side by side, for even when it is possible to get a differential equation for the displacement of the sphere, it appears that we cannot determine the constants of integration without a knowledge of the state of the medium.

I should like to acknowledge my great obligation to Prof. Love for his kindness in criticising this paper. On his suggestion some portion of it has been rewritten, and the results thereby represented in a more satisfactory manner.
